On Chiral Symmetry Breaking in a Non-Simply Connected Space-Time

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We consider QCD in $R^3 \times S^1$ and show that non-trivial global space-time topology breaks chiral symmetry.

Quantum field theories are affected by local as well as global characteristics of space-time. Local characteristics such as curvature affect field equations. Likewise, global characteristics can also induce significant changes in the theory. Studies on the effect of global characteristics on quantum field theory were initiated by Isham [1], and later in a series of papers [2–4] the working rules were set up. It was shown that in certain spaces M with non-trivial topology there are new field configurations and that the number of new field configurations is equal to the number of elements of the first cohomology group H^1 (M, Z_2) of the space M. We shall be working in the space $M = R^3 \times S^1$. The choice of this space is dictated by its relative simplicity. This space admits a flat metric and it is non-simply connected. Also, there exist two spin configurations in this space. These are called untwisted and twisted fields and they obey respectively periodic and anti-periodic boundary conditions. In this note we shall consider a SU(2) QCD theory in $R^3 \times S^1$ and it will be shown that the quarks gain different (topological) mass resulting in chiral symmetry breaking.

Before taking up the question of chiral symmetry breaking, first of all it is necessary to present the $R^3 \times S^1$ formalism:

$$0 \leqq x^1 \leqq L\,, \quad -\infty < x^0, x^2, x^3 < +\infty\,, \tag{1}$$

with the condition that the ends of the x^1 axis are identified.

$$p_1 \to \frac{2\pi m}{L} \qquad \text{for untwisted fields,}$$

$$p_1 \to \frac{(2m+1)\pi}{L} \qquad \text{for twisted fields,}$$

$$m = 0, \pm 1, \pm 2, \dots \tag{2}$$

$$\int \frac{d^4 p}{(2\pi^4)} \to \frac{(2\pi)}{L} \sum_m \int \frac{d^3 p}{(2\pi)^3}\,. \tag{3}$$

The general form of the propagators for the spinor and vector boson fields are given, respectively, by

$$S_{ab} = i\,\delta_{ab}\,\hat{p}/(p^2 + i\varepsilon)\,, \tag{4}$$

$$D^{ab}_{\mu\gamma} = -\,i\,g_{\mu\gamma}\,\delta_{ab}/(p^2 + i\varepsilon)\,, \tag{5}$$

$$\hat{p} = p_\mu \cdot \gamma^\mu\,. \tag{6}$$

The Lagrangian is given by

$$L = -\frac{1}{4}(F^a_{\mu\nu})^2 + i\sum_{j=1}^{2} \bar{\psi}_j \not{D}\, \psi_j\,, \tag{7}$$

where the sum is carried over the two quark flavours, and these flavours are identified with twisted and untwisted field configurations.

It is our purpose to show that the non-trivial global space-time topology lifts the mass degeneracy between the quarks and as such is necessary to calculate the quark self-energy to second order in the coupling constant (Figure 1).

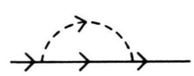

Fig. 1. Quark Self-energy

The quark self-energy is given by

$$(-i)\sum(k,n) = \frac{(-ig)^2}{2L}\sum_m \int \frac{d^n p}{(2\pi)^n} \cdot \gamma^\mu \frac{i}{\hat{p} + \hat{k} + i\varepsilon}\,\gamma^\nu \frac{(-i)\,g_{\mu\nu}}{p^2 + i\varepsilon}\,. \tag{8}$$

Now, using the results

$$\int \frac{d^n x}{(x^2 + a)^\alpha} = \frac{\pi^{n/2}\,\Gamma(\alpha - n/2)\,a^{n/2-\alpha}}{\Gamma(\alpha)}\,, \tag{9}$$

$$\int \frac{p^\mu \cdot d^n p}{(p^2 + a)^\alpha} = 0\,, \tag{10}$$

$$\frac{1}{ab} = \int_0^1 dx\,[(a-b)\,x + b]^{-2}\,, \tag{11}$$

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we obtain from (8)

$$\sum (0, n) = \frac{3}{4} \frac{\gamma^1 (1-n)\, \Gamma\,(2-n/2)\, \pi^{n/2} g^2}{4\,(2\pi)^n\, L\, \Gamma\,(2)} \tag{12}$$

$$\cdot \sum_m (m+a) \int_0^1 [(2\,m\,a + a^2)\, x + m^2]^{n/2-2}\, \mathrm{d}x\,,$$

where $a = 0$ or ½ depending on wheather the quark is untwisted or twisted. From (12) it is also clear that, had we choosen to work in Minkowski space R^4, the singularity of (8) would have appeared as pole of the gamma function, but in the present case ($n = 3$) the singularity appears in the summation.

Now, consider the case of a twisted quark. In this case

$$\sum (0, 3) = -\gamma^1 \frac{3g^2}{16\pi L} \cdot \sum_m \frac{m+1/2}{m+1/4} \tag{13}$$

$$= -\gamma \frac{3g^2}{16\pi L} \cdot \frac{1}{4} \left[\sum_0^\infty \frac{1}{(m+1/4)} - \sum_0^\infty \frac{1}{(m+3/4)} + 8 \sum_0^\infty 1 \right]. \tag{14}$$

The r.h.s. of (14) is clearly divergent. To regularize it, we consider the expression [5]

$$\sum_0^\infty \frac{1}{(m+1/4)^s} - \sum_0^\infty \frac{1}{(m+3/4)^s}$$

$$= \zeta\,(s, \tfrac{1}{4}) - \zeta\,(s, \tfrac{3}{4})\,, \tag{15}$$

where $\zeta\,(s, a)$ denotes the generalized Riemann Zeta function (cf. [6]).

Next using the relation [6]

$$\lim_{S\to 1} \left[\zeta\,(S, a) - \frac{1}{S-1} \right] = -\,\psi\,(a)\,, \tag{16}$$

$$\zeta\,(z) = \lim_{\alpha\to 0}\, [\zeta\,(z, \alpha) - \alpha^{-z}]\,. \tag{17}$$

We obtain from (14)

$$\sum (0, 3) \overset{R}{=} -\gamma^1 \frac{3g^2}{16} (\pi/4 - 1)\,, \tag{18}$$

where $\overset{R}{=}$ indicates the regularized value.

Hence the twisted quark mass is given by

$$m_T = \gamma^1 \sum (0, 3) = \frac{3\,g^2\,(\pi/4 - L)}{16\,\pi\,L}\,. \tag{19}$$

Proceeding similarly it can be shown that the untwisted quark mass vanishes

$$m_{u\,T} = 0\,. \tag{20}$$

From (19) and (20) it is seen that the quarks have obtained different topological masses, and thus chiral symmetry is broken.

In conclusion, it has been shown that global space-time topology lifts the mass degeneracy between the quarks (originally massless) and thus produces chiral symmetry breaking. We note that, though our conclusion agrees with Goncharov [7], the essential details are totally different (because of a faulty regularization procedure adopted by Goncharov). Finally we advance some reasons for considering field theory in $R^3 \times S^1$: (i) it is necessary to consider curved space-time to describe the early universe, (ii) space-time structure in the scale of 1 fermi or less is unknown.

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